# SOLUTIONS WITHOUT DUMMY AXIOM FOR TU COOPERATIVE GAMES

L. HERNÁNDEZ LAMONEDA, R. JUÁREZ GARCÍA, AND F. SÁNCHEZ SÁNCHEZ

Abstract

In this paper we study all additive, symmetric and efficient solutions, i.e., the set of axioms that traditionally are used to characterize the Shapley value except the dummy axiom. Also we obtain an expression for this kind of solutions when we also include the self dual axiom. These expressions allow us to give an alternative form to the consensus value, the generalized consensus value and the solidarity solution. Furthermore, we introduce a new axiom called coalitional independence to replace the symmetry axiom in order to get similar results.

*Date*: June, 2005.

CIMAT

## 1. Preliminaries

**Definition 1.** *By a game we mean a pair $(N, v)$ where $N \subset \mathbb{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a real function such that $v(\emptyset) = 0$. Let $G = G^N$ be the set of games with a fixed set of players $N$.*

We consider $N$ fixed and $n = |N|$.

Let $(N, z)$ be the zero game, i.e., the game defined by $z(T) = 0$ for every $T \subseteq N$.

**Definition 2.** *By a solution in $G$ we mean a continuous function $\varphi : G^N \to \mathbb{R}^N$ such that $\varphi(z) = 0$.*

CIMAT

**Axiom 1** (Additivity). *The solution $\varphi$ is additive if* $\varphi(v+w) = \varphi(v) + \varphi(w)$ *for every* $v, w \in G^N$.

**Axiom 2** (Efficiency). *The solution $\varphi$ is efficient if* $\sum_{i \in N} \varphi_i(v) = v(N)$ *for every* $v \in G^N$.

**Axiom 3** (Dummy). *If the player $i$ is a dummy player in $(N, v)$ then* $\varphi_i(v) = 0$.

**Axiom 4** (Symmetry). *The solution $\varphi$ is said to be symmetric if and only if* $\varphi(\theta^* v) = \theta^* \varphi(v)$ *for every $\theta$ and* $v \in G$.

CIMAT

**Theorem 1** (Shapley, 1953)**.** *There exists a unique solution* $\varphi$ *that satisfies additivity, symmetry, dummy and efficiency axioms. Furthermore it is given by*

$$\varphi_i(v) = \sum_{S \not\ni i} \frac{s!(n-s-1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

## 2. Solutions without a dummy axiom

### 2.1. General expression.

**Proposition 1.** *The solution $\varphi$ satisfies additivity, symmetry and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N \backslash S)] \tag{2.1}$$

*for some $n-1$ real numbers $\{\beta_s\}_{s=1}^{n-1}$.*

## 2.2. Interpretation.

Step 1.

$$\frac{v(N)}{n} \to \text{player } i$$

Step2.

For every $S \neq N$ we transfer $(n-s)s\beta_s v(S)$ from $N \backslash S$ to $S$ as follows,

Each player in $N \backslash S$ pays:

$$s\beta_s v(S)$$

Each player in $S$ receives:

$$(n-s)\beta_s v(S)$$

At the end, the player $i$ has the amount $\varphi_i(v)$ given by (2.1).

CIMAT

2.3. **Random Process.**

$\mathbf{S}$ random variable

Selection of $S$ :

**a)** $P(\mathbf{S}=N)=\frac{1}{n}$

- Each player receives:

$$v(N)$$

**b)** For $S \neq N$, $P(\mathbf{S}=S)=\beta_s$ (same number for coalitions with same cardinality)

- Each player in $N \backslash S$ pays:

$$sv(S)$$

- Each player in $S$ receives:

$$(n-s)v(S)$$

The expected value that player $i$ gets in this process is equal to the amount $\varphi_i(v)$ given by (2.1).

CIMAT

## 2.4. Self Dual property.

**Definition 3.** *Given a game $(N, v) \in G$, we say that $v^*$ is its dual game if*

$$v^*(S) = v(N) - v(N \backslash S)$$

*for every $S \subseteq N$.*

**Axiom 5** (Self dual)**.** *We say that the solution $\varphi$ is self dual if $\varphi(v) = \varphi(v^*)$ for every game $v \in G$.*

**Corollary 1.** *The solution $\varphi$ satisfies additivity, symmetry, efficiency and self dual axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_s v(S) - \beta_{n-s} v(N \backslash S)] \tag{2.2}$$

*for some set of $[\frac{n+1}{2}]$ real numbers $\{\beta_s\}_{s=1}^{n-1}$ such that $\beta_s = \beta_{n-s-1}$.*

CIMAT

2.5. **Kernel of a solution.** The kernel of a solution $\varphi$ is the vector space of games $v$ such that $\varphi(v) = 0$.

Let $w_T$ be given by

$$w_T(S) = \begin{cases} 1 & \text{if } S = \{j\},\, j \notin T \\ \frac{\beta_1}{\beta_t} & \text{if } S = T \\ 0 & \text{otherwise} \end{cases}$$

for every $T \subset N$, $T \neq N$ and $|T| \geq 2$. Furthermore, let $w_N$ be

$$w_N(S) = \begin{cases} 1 & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 2.** *Let $\varphi$ be given by (2.1) such that $\beta_t \neq 0$ for $t = 1, \ldots, n-1$, then the set of games $\{w_T\}_{|T|\geq 2}$ form a basis for the kernel of $\varphi$.*

CIMAT

## 3. Solutions with the coalitional independence axiom

**Definition 4.** *We say that the two games* $(N, v)$ *and* $(N, w)$ *only differ in* $S$ *if and only if* $v(T) = w(T)$ *for every coalition* $T \neq S$.

**Axiom 6** (Coalitional independence)**.** *We say that* $\varphi$ *satisfies the coalitional independence axiom if*

$$\varphi_i(v) - \varphi_i(w) = \varphi_j(v) - \varphi_j(w)$$

*for every two games* $(N, v)$ *and* $(N, w)$ *that only differ in* $S$ *and* $i$, $j \in S$ *or* $i$, $j \in N \backslash S$.

CIMAT

**Remark 2.** *Additivity and symmetry imply coalitional independence, but additivity and coalitional independence do not imply symmetry.*

**Proposition 3.** *The solution $\varphi$ satisfies additivity, coalitional independence and efficiency axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N \setminus S} v(N \setminus S)] \tag{3.1}$$

*for some set of* $2^n - 2$ *real numbers* $\{\beta_S\}_{\emptyset \neq S \subsetneq N}$.

CIMAT

**Corollary 2.** *The solution $\varphi$ satisfies additivity, coalitional independence, efficiency and self dual axioms if and only if it is of the form*

$$\varphi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} (n-s)[\beta_S v(S) - \beta_{N \backslash S} v(N \backslash S)] \tag{3.2}$$

*for some set of* $2^{n-1}-1$ *real numbers* $\{\beta_S\}_{S \subseteq N}$ *such that* $\beta_S = \beta_{N \backslash S}$.

**Proposition 4.** *The Shapley value is the unique solution that satisfies additivity, coalitional independence, dummy and efficiency axioms.*

CIMAT

## 4. Some special cases

A first example is the Equal Surplus solution:

$$\varphi_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n}$$

that we get when we replace $\beta_1 = \frac{1}{n}$ and $\beta_s = 0$ otherwise in (2.1).

Another self dual solution with a simple expression is

$$\varphi_i(v) = \sum_{S \ni i} \frac{v(S)}{s} - \sum_{S \not\ni i} \frac{v(S)}{n-s}$$

where $\beta_s = \frac{1}{s(n-s)}$ in (2.1).

CIMAT

**4.1. The Consensus Value.** We get the convex linear combination of two solutions of the form (2.1) by the corresponding convex linear combinations of their parameters:

$$(1-\theta)\varphi^{\beta}+\theta\varphi^{\gamma}=\varphi^{(1-\theta)\beta+\theta\gamma}.$$

Moreover, Ju et al. [3] prove that the consensus value is the middle point between the Equal Surplus solution and the Shapley value. So, we get an expression for the consensus value:

$$\frac{v(N)}{n}+\frac{1}{2}\left[v(\{i\})-\frac{\sum_{k\neq i}v(\{k\})}{n-1}\right]+\sum_{S\ni i,|S|\neq n,n-1,1}(n-s)[\beta_s v(S)-\beta_{n-s}v(N\backslash S)].$$

In the same way, we could generate an expression for any generalized consensus value, i.e., we only need replace $\beta_1=\frac{1-\theta}{n}+\frac{\theta}{n(n-1)}$ and $\beta_s=\frac{\theta(s-1)!(n-s-1)!}{n!}$ for $s=2,..,n-1$, in (2.1).

CIMAT

4.2. **Solidarity value.** Nowak and Radzik [5] introduce the solidarity value. They define, for any non-empty coalition $T$ and any game $v \in G$,

$$A^v(T) = \frac{1}{t}\sum_{k\in T}[v(T) - v(T\backslash\{k\})]$$

Then, they define the solidarity value for player $i$ as,

$$\psi_i(v) = \sum_{T\ni i}\frac{(n-t)!(t-1)!}{n!}A^v(T) \tag{4.1}$$

They characterized this value with the efficiency, additivity, symmetry and A-null player axioms, so the solidarity value must be a special case of (2.1). Indeed, if we expand (4.1) we get that the coefficient of $v(S)$, for a coalition $T$ which does not contain $i$, is $\frac{(n-s-1)!s!}{n!}\frac{1}{s+1}$. Thus, this

CIMAT

coefficient correspond to $s\beta_s$ in (2.1), and therefore

$$\beta_s = \frac{(n-s-1)!(s-1)!}{(s+1)n!}$$

what give us an alternative expression of (4.1):

$$\psi_i(v) = \frac{v(N)}{n} + \sum_{S \ni i, S \neq N} \frac{(n-s)!(s-1)!}{n!} \left[ \frac{v(S)}{s+1} - \frac{v(N \backslash S)}{n-s+1} \right].$$

Moreover, the solidarity value it is not self dual since $\beta_s \neq \beta_{n-s}$.

CIMAT

4.3. **Least Square Prenucleolus.** Lastly, Ruiz, L.M., Valenciano, F. and Zarzuelo J.M. [6] introduce the Least Square Prenucleolus solution,

$$\lambda_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}} \left[ \sum_{S \ni i} (n-s)v(S) - \sum_{S \not\ni i} sv(S) \right]$$

This solution is also of the form (2.1). The corresponding parameters are $\beta_s = \frac{1}{n2^{n-2}}$.

## References


[1] Dubey P., Neyman A. and Weber R.J. (1981). "Value Theory without Efficiency", *Mathematics of Operations Research*, Vol.6, No.1, pp. 122-128.

[2] Chun, Y. (1989). "A New Axiomatization of the Shapley Value," *Games and Economic Behavior*, Vol.1, Num. 2, pp.119-130.

[3] Ju Y., Borm P. and Ruys P. (2004) "The Consensus Value: a New Solution Concept for Cooperative Games," *CentER Discussion Paper* No. 2004-50.

[4] Jurez R., Herndez L., and Snchez F.. Dissection of cooperative solutions in game theory using representation techniques. (Submitted)

[5] Nowak A. S. and Radzik T. (1994). " A Solidarity Value for $n$-Person Transferable Utility Games," *International Journal of Game Theory*, Vol. 23, pp.43-48.

[6] Ruiz, L.M., Valenciano, F. and Zarzuelo J.M. (1996) “The least Square Prenucleoulus and the Least Square Nucleolus. Two values for TU Games Based on teh Excess Vector”, *International Journal of Game Theory*,Vol. 25, pp113-134.

[7] Shapley L.S. (1953). A value for $n$-person games , in “*Contributions to the Theory of Games II*,” pp. 307-312, Annals of Mathematics Studies Vol. 28, Princeton University Press, Princeton.